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Equivalence Transformation for the Filtration Problem

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Abstract—In this paper, I am considering a filtration equation, and by using Ovsiannikov's method, I intend to construct the most general equivalence algebra. Moreover, some algorithms are performed in order to extend the principal algebra.

Keywords—Dimensional motion of a gas, Group classification, Similarity solutions.

1. INTRODUCTION

We deal with the nonlinear diffusion equation as follows:

$$h_t = k_1 \left\{ h h_{xx} + h h_{yy} + h_x^2 + h_y^2 \right\} + k_2 g(t, x, y, h), \tag{1.1}$$

where $\mathbf{k}_1, \mathbf{k}_2$ are constants and g is an arbitrary function of its arguments.

The study of equation (1.1) is stimulated not only by physical examples mentioned in [1] but also by other examples: nonlinear nonhomogeneous diffusion equations.

In this paper, we are looking for the extensions of the principal Lie algebra admitted by a class of differential equations among the elements of its equivalence algebra. For such problems of type (1.1), a new method termed preliminary group classification was proposed. This method reduces the problem of classification to the algebraic problem of constructing the optimal system of subalgebra of the equivalence Lie algebra. However, in general, there are no effective algorithms for constructing the optimal system of an infinite-dimensional algebra.

Section 2 is devoted to constructing the equivalence algebra for equation (1.1), which allows us to obtain the principal Lie algebra.

In Section 3, we apply the method of preliminary group classification [2,3] in order to obtain extensions of principal Lie algebra. It is worthwhile to notice that these results allow us to obtain only a partial classification and we observe that we can easily obtain some of the results by such simple calculation. In contrast, using Lie method for symmetries usually leads to determining equations which are difficult to solve.

2. EQUIVALENCE ALGEBRA

An equivalence transformation of equation (1.1), according to our case, is leaving invariant the equation under consideration. This means that any equation of the form (1.1) is mapped by

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an equivalence transformation into an equation of the same form. In the following operator, we consider the generators of the continuous group of equivalence transformation in the form

$$\tilde{Y} = \zeta_1 \frac{\partial}{\partial t} + \zeta_2 \frac{\partial}{\partial x} + \zeta_3 \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial h} + \mu \frac{\partial}{\partial g}, \tag{2.1}$$

where the coordinates ζ_1, ζ_2, ζ_3, and η of the operator (2.1) are sought as a function of t, x, y, and h, while the coordinate μ is sought as a function of t, x, y, h, and g. The second prolongation of $\tilde{Y}$ which we need is in the form:

$$Y = \tilde{Y} + \eta^t \frac{\partial}{\partial u_t} + \eta^x \frac{\partial}{\partial u_x} + \eta^y \frac{\partial}{\partial u_y} + \eta^{xx} \frac{\partial}{\partial u_{xx}} + \eta^{yy} \frac{\partial}{\partial u_{yy}}, \tag{2.2}$$

where η^t, η^x, η^y, η^{xx}, and η^{yy} are given [4,5].

Considering the invariance (1.1) with respect to the operator $\tilde{Y}$, we obtain a linear PDE, a system usually called the determining system. And by using a well-known procedure [6,7], we obtain

$$\begin{gathered} \zeta_1 = \alpha_1 t + \alpha_0, \qquad \zeta_2 = \beta_1 x + \beta_2 y + \beta_0, \\ \zeta_3 = -\beta x + \beta_1 y + \gamma_0, \qquad \eta = (2\beta_1 - \alpha_1) u, \\ \mu = 2\beta_1 g, \end{gathered} \tag{2.3}$$

where α_0, α_1, β_0, β_1, β_2, and γ_0 are arbitrary constants.

The equivalence algebra $L_{\mathfrak{R}}$ is finite-dimensional and spanned by

$$\begin{gathered} Y_1 = \partial_t, \qquad Y_2 = \partial_x, \qquad Y_3 = \partial_y, \\ Y_4 = t\partial_t - u\partial_u, \\ Y_5 = x\partial_x + y\partial_y = 2u\partial_u + 2g\partial_g, \\ Y_6 = y\partial_x - x\partial_y. \end{gathered} \tag{2.4}$$

The most general symmetry algebra (1.1) for arbitrary g is said to be principal Lie algebra [6]. This principal Lie algebra can be easily found by projecting the equivalence algebra according to the following proposition.

PROPOSITION 1. *An operator X belongs to the principal Lie algebra $\mathbf{L}_p$ for equation (1.1) iff*

$$X = \mathrm{Pr}_{(t,x,y,h)} \left(\tilde{Y}\right), \tag{2.5}$$

with an equivalence generator $\tilde{Y}$, such that

$$\mathrm{Pr}_{(h,g)} \left(\tilde{Y}\right) = 0. \tag{2.6}$$

PROPOSITION 2. *Letting $\tilde{Y}$ be an equivalence operator, the operator*

$$X = \mathrm{Pr}_{(t,x,y,h)} \left(\tilde{Y}\right) \tag{2.7}$$

is a symmetry operator for equation (1.1) with function $g(t,x,y,h)$ iff the $\mathbf{g}(\mathbf{t},\mathbf{x},\mathbf{y},\mathbf{h})$ is invariant under the group generated by

$$Z = \mathrm{Pr}_{(t,x,y,h)} \left(\tilde{Y}\right). \tag{2.8}$$

The proof of these statements appears in [3]. Using equations (2.4) and Proposition 1, therefore, the principal Lie algebra is three-dimensionally spanned by

$$\begin{gathered} X_1 = \partial t, \qquad X_2 = \partial y, \\ X_3 = y\partial_x - x\partial_y. \end{gathered} \tag{2.9}$$

3. ON THE PRELIMINARY GROUP CLASSIFICATION

The method of the preliminary group classification was introduced first [8]. This method consists in the classification of all nonequivalent equations with respect to a given equivalence group G_ε admitting ε-extensions of the principal Lie algebra. G_ε is not necessarily the largest equivalence group, but it can be any subgroup of all the equivalence transformation groups.

Then the nonzero projections of (2.8) on the space (t, x, y, h, g) are

$$\begin{gathered} Z_1 = \partial, \qquad Z_2 = \partial_x, \qquad Z_3 = \partial_y, \\ Z_4 = t\partial_t - u\partial_u, \qquad Z_5 = x\partial_x - y\partial_y - 2j\partial_u + 2g\partial_g, \\ Z_6 = y\partial_x - x\partial_y, \end{gathered} \tag{3.1}$$

where

$$\begin{aligned} &Z_1 = \mathrm{pro}(Y_1), \qquad Z_2 = \mathrm{pro}(Y_2), \qquad Z_2 = \mathrm{pro}(Y_2), \\ &Z_4 = \mathrm{pro}(Y_4), \qquad Z_5 = \mathrm{pro}(Y_5), \qquad Z_6 = \mathrm{pro}(Y_6). \end{aligned} \tag{3.2}$$

We denote by L_6 the Lie algebra spanned by operators (3.2) whose table of commutators is the following.

Table 1.

	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6
Z_1	0	0	0	Z_1	0	0
Z_2	0	0	0	0	Z_2	Z_3
Z_3	0	0	0	0	Z_3	Z_2
Z_4	Z_1	0	0	0	0	0
Z_5	0	Z_2	Z_3	0	0	0
Z_6	0	Z_3	Z_2	0	0	0

The problem of preliminary group classification of equation (1.1) lies in how to construct the nonsimilar subalgebras of L_6, or optimal systems of subalgebras.

Adjoint Group For Algebra L_6

We need the adjoint algebra L_6 and the corresponding adjoint group. Fixed $\alpha = 1, 2, 3, 4, 5, 6$. The elements A of the adjoint algebra L_6 are

$$\mathbf{A}_\alpha = [\mathbf{Z}_\alpha, \mathbf{Z}_\beta] \frac{\partial}{\partial_{\mathbf{Z}_\beta}}, \qquad \beta = 1, 2, 3, 4, 5, 6.$$

As a basis of the algebra adjoint L_6

$$\begin{aligned} &A_1 = Z_1\partial_{Z_4}, &&A_2 = Z_2\partial_{Z_5} - Z_3\partial_{Z_6}, \\ &A_3 = Z_3\partial_{Z_5} + Z_2\partial_{Z_6}, &&A_4 = -Z_1\partial_{Z_1}, \\ &A_5 = -Z_2\partial_{Z_2} - Z_3\partial_{Z_3}, &&A_6 = Z_3\partial_{Z_2} - Z_2\partial_{Z_3}, \end{aligned}$$

which generates the following one-parameter groups of linear transformations:

$$Z'_\beta = Z_\beta + a_\alpha[Z_\alpha, Z_\beta], \qquad \beta = 0, 1, \ldots, 6.$$

The infinitesimal operator A_1 generates the following one-parameter group of linear transformation:

$$\begin{aligned} &Z'_1 = Z_1, &&Z'_2 = Z_2, \\ &Z'_3 = Z_3, &&Z'_4 = Z_4 + \alpha_1 Z_1, \\ &Z'_5 = Z_5, &&Z'_6 = Z_6, \end{aligned}$$

which is represented by the matrix

$$M_1(a_1) = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix}, \qquad -\infty < a_1 < \infty.$$

Following the same procedure, we obtain the matrices $M_2(a_2), \dots, M_6(a_6)$ as associated to the infinitesimal operators $A_2, \dots, A_6$, respectively. Here $0 < a_5, a_6 < +\infty$, $-\infty < a_1, a_2, a_3, a_4 < +\infty$. We save the writing of these matrices because we need only their product for our results:

$$M = \prod_{\alpha} M_\alpha(a_\alpha) = \begin{Vmatrix} 1-a_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-a_5 & a_6(1-a_5) & 0 & 0 & 0 \\ 0 & 0 & -a_6(1-a_5) & 0 & 0 & 0 \\ a_1(1-a_4) & 0 & 0 & 1 & 0 & 0 \\ 0 & a_2 & a_3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix}.$$

The linear transformations associated to the matrix M allow us to obtain the collection of pairwise nonconjugate one-dimensional subalgebras of $\mathbf{L}_5$. It is preferable not to work with the operators $Z_1, Z_2, \dots, Z_6$, but with coordinates of the decomposition

$$Z = \sum_{i=1}^{6} e^i Z_i \tag{3.3}$$

of $Z \in L_6$, i.e, with the vectors

$$e = \left(e^1, e^2, e^3, e^4, e^5, e^6\right).$$

Vector $\mathbf{e}$ is transformed by means of the transposed matrix $\mathbf{M}^\top$ of $\mathbf{M}$ in the case that the transformation has the following coordinates:

$$\begin{aligned} e^{-1} &= (1-a_4)e^1 + a_1(1-a_4)e^4, \\ e^{-2} &= (1-a_5)e^2 + a_2 e^5, \\ e^{-3} &= a_6(1-a_5)e^2 - a_6(1-a_5)e^3 + a_3 e^5, \\ e^{-4} &= e^4, \\ e^{-5} &= e^5, \\ e^{-6} &= e^6. \end{aligned} \tag{3.4}$$

So, the construction of the optimal system of two-dimensional subalgbras of L^6 follows in a very natural way. Obviously we choose a representative of each nonequivalence class which has the simplest form. After some calculation, we obtain the following optimal system subalgebra of L^6:

$$\alpha Z_4 + Z_6, \qquad Z_3 + Z_6 + \alpha Z_4, \qquad Z_3 + Z_6, \tag{3.5}$$

where α is an arbitrary real number.

Applying Proposition 2 to the operator (3.5) results in the nonequivalent equation (1.1) admitting ε-extensions of the principal Lie algebra L_3 by one, i.e., equations of the form (1.1) admit a four-operator X_4, together with the three basis operators (2.9). For every case, when this extension occurs, we indicate the corresponding coefficients g and the additional operator X_4. Then the results are summarized as the following.

1. The operator $Z_3 + Z_6$ equation is

$$h_t = k_1\left\{hh_{xx} + hh_{yy} + h_x^2 + h_y^2\right\} + k_2 H(\xi_1, \xi_2, \xi_3),$$

where

$$\xi_1 = \frac{1}{2}y^2 + \frac{1}{2}x^2 - x, \qquad \xi_2 = t, \qquad \xi_3 = h,$$

and the additional operator is

$$X_4 = (1-x)\partial_y + y\partial_x.$$

2. For $\alpha Z_4 + Z_6$ ($\alpha \neq 0$), the equation is

$$h_t = k_1\left\{hh_{xx} + hh_{yy} + h_x^2 + h_y^2\right\} + k_2 H(\xi_1, \xi_2, \xi_3),$$

and

$$\xi_1 = x^2 + y^2, \qquad \xi_2 = t^{1/\alpha} e^{\sin^{-1} y/\sqrt{\xi_1}}, \qquad \xi_3 = \frac{t}{h},$$

then the additional operator is

$$X_4 = \alpha t\partial_t + y\partial_x - x\partial_y.$$

3. When the operator $Z_3 + Z_6 + \alpha Z_4$ equation is

$$h_t = k_1\left\{hh_{xx} + hh_{yy} + h_x^2 + h_y^2\right\} + k_2 H(\xi_1, \xi_2, \xi_3),$$

and

$$\xi_1 = \frac{1}{2}y^2 + \frac{1}{2}x^2 - x, \qquad \xi_2 = t^{1/\alpha} e^{\sinh^{-1}(x+1)/\sqrt{1-2\xi_1}}, \qquad \xi_3 = th,$$

then the additional operator is

$$X_4 = (1-x)\partial_y + y\partial_x + \alpha t\partial_t.$$

4. CONCLUSION

It is worthwhile to note that these results give not only a partial classification, but also allow us to obtain the principal Lie algebra. When the physical problem gives a specialization of the form of the function g, we also obtain all nonequivalent equations (1.1) meeting ε-extensions of the principal Lie algebra L_3 by one, i.e., a fourth operator X_4 is admitted by equations of the formula (1.1) together with the three basic operators (2.9) of L_3. For every case, when this extensions occurs, we intact the corresponding coefficient g and the additional operator X_4.

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